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LETTER TO THE EDITOR

A lattice spanning-tree entropy function

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Abstract
The function

$$W(a,b) = \int_0^{2\pi} \mathrm{d}x \int_0^{2\pi} \mathrm{d}y \ln[1 - a\cos x - b\cos y - (1-a-b)\cos(x+y)]$$

which expresses the spanning-tree entropy for various two-dimensional lattices, for example, is evaluated directly in terms of standard functions. It is applied to derive several limiting values of the triangular lattice Green function.

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Introduction

The function

$$W(a,b) = \int_0^{2\pi} \mathrm{d}x \int_0^{2\pi} \mathrm{d}y \ln[1 - a\cos x - b\cos y - c\cos(x+y)]$$

with $a + b + c = 1$ arises frequently in the statistical physics and combinatorics of two-dimensional lattice systems. For example:

(i) $a = b = 1/2$

$$S_{sq} = \frac{\ln(2)}{2\pi^2} + \frac{W(a,b)}{4\pi^2}$$

is the spanning-tree entropy for the square lattice [1].

(ii) $a = b = 1/3$

$$S_{tr} = \frac{\ln(6)}{4\pi^2} + \frac{W(a,b)}{4\pi^2}$$

is the spanning-tree entropy for the triangular lattice [2].

(iii)

$$a = \frac{\sinh K_1}{\sinh K_1 + \sinh K_2 + \sinh K_3}$$
$$b = \frac{\sinh K_2}{\sinh K_1 + \sinh K_2 + \sinh K_3}$$
$$F_I = \ln(2) + \frac{1}{8\pi^2}\ln[\sinh K_1 + \sinh K_2 + \sinh K_3] + \frac{W(a,b)}{8\pi^2}$$

is the critical free energy of the Ising model on a triangular lattice [3].

By comparing the free energies of various Potts models which are known to be related and have been worked out in different ways, Chen and Wu [4] have proposed that

$$\frac{1}{4\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \ln[A + B + C - A\cos\theta - B\cos\phi - C\cos(\theta+\phi)]$$
$$= -\ln(2S) + \frac{2}{\pi}[Ti_2(AS) + Ti_2(BS) + Ti_2(CS)]$$

where $A, B, C \geqslant 0$ and $S = 1/\sqrt{AB + BC + CA}$. The aim of this letter is to provide a direct proof of this formula.

Calculation

By symmetry, one easily finds

$$W(a,b) = 2[W_+(a,b) + W_-(a,b)] \tag{1}$$

where

$$W_\pm(a,b) = \int_0^\pi \mathrm{d}x \int_0^\pi \mathrm{d}y \ln[1 - a\cos x - b\cos y - c\cos(x \pm y)].$$

Next, we make the standard change of variable

$$u = \tan(x/2) \qquad v = \tan(y/2)$$

to obtain

$$W_+(a,b) = 4\int_0^\infty\int_0^\infty \frac{\mathrm{d}u\,\mathrm{d}v}{(1+u^2)(1+v^2)} \ln\left[\frac{2}{(1+u^2)(1+v^2)}\right] + 4\int_0^\infty\int_0^\infty \ln[(u+v)^2 - a((u+v)^2 - u^2(1+v^2)) - b((u+v)^2 - v^2(1+u^2))]\frac{\mathrm{d}u\,\mathrm{d}v}{(1+u^2)(1+v^2)}.$$

The first integral is elementary, giving

$$W_+(a,b) = -3\pi^2\ln(2) + 4\int_0^\infty\int_0^\infty \frac{\ln[c(u+v)^2 + au^2(1+v^2) + bv^2(1+u^2)]}{(1+u^2)(1+v^2)}\,\mathrm{d}u\,\mathrm{d}v.$$

Similarly,

$$W_-(a,b) = -3\pi^2\ln(2) + 4\int_0^\infty\int_0^\infty \frac{\ln[c(u-v)^2 + au^2(1+v^2) + bv^2(1+u^2)]}{(1+u^2)(1+v^2)}\,\mathrm{d}u\,\mathrm{d}v.$$

By inserting the last two expressions into (1) and noting that the resulting integrand is even in v, we have

$$W(a,b) = -12\pi^2\ln(2) + 8F(a,b),$$

where

$$F(a,b) = \int_0^\infty \frac{\mathrm{d}u}{1+u^2}\int_{-\infty}^\infty \frac{\mathrm{d}v}{1+v^2}\ln[(1-b)u^2 + (1-a)v^2 + (a+b)u^2v^2 + 2cuv].$$

Next, let $u = vw$ to obtain

$$F(a,b) = 2\int_0^\infty \frac{u\,\mathrm{d}u}{1+u^2}\ln(u)\int_{-\infty}^\infty \frac{\mathrm{d}w}{1+u^2w^2} + \int_0^\infty \frac{\mathrm{d}u}{u(1+u^2)}\int_{-\infty}^\infty \frac{\mathrm{d}w}{w^2+u^{-2}}$$
$$\times \ln[1-b+(1-a)w^2+(a+b)u^2w^2+2cw].$$

The first integral vanishes and since [5]

$$\int_0^\infty \frac{\mathrm{d}w}{w^2+d^2}\ln[\alpha w^2+2\beta w+\gamma] = \frac{\pi}{d}\ln\left[\alpha d^2+\gamma+2d\sqrt{\alpha\gamma-\beta^2}\right],$$

after the substitution $u = 1/z$,

$$F(a,b) = \pi\int_0^\infty \frac{\mathrm{d}z}{1+z^2}\ln\left[(1-a)z^2+(1+a)+2\sqrt{[ac+b(1-b)]z^2+(1-b)(a+b)}\right]. \qquad (2)$$

Now, let us define

$$A = \frac{y\cot(\theta/2)}{1+\sqrt{1+y^2}}, \qquad B = \frac{y\tan(\theta/2)}{1+\sqrt{1+y^2}}.$$

Then $(A+B)/(1-AB) = y\csc\theta$, so $\tan^{-1}(y\csc\theta) = \tan^{-1}A + \tan^{-1}B$. However,

$$\csc\theta\tan^{-1}(y\csc\theta) = -\frac{\mathrm{d}}{\mathrm{d}\theta}\int_1^{\csc\theta}\frac{\tan^{-1}(yu)}{\sqrt{u^2-1}}\,\mathrm{d}u$$
$$\csc\theta\tan^{-1}A = -\frac{\mathrm{d}}{\mathrm{d}\theta}\int_0^A \frac{\tan^{-1}u}{u}\,\mathrm{d}u$$
$$\csc\theta\tan^{-1}B = \frac{\mathrm{d}}{\mathrm{d}\theta}\int_0^B \frac{\tan^{-1}u}{u}\,\mathrm{d}u.$$

Hence, since both sides vanish for $\theta = \pi/2$,

$$\int_1^{\csc\theta}\frac{\tan^{-1}(yu)}{\sqrt{u^2-1}}\,\mathrm{d}u = Ti_2(A) - Ti_2(B) \qquad (3)$$

where

$$Ti_2(z) = \int_0^z \frac{\tan^{-1}x}{x}\,\mathrm{d}x.$$

With

$$\theta = \csc^{-1}\sqrt{\frac{b^2-a^2+1}{1-a^2}}, \qquad y = \sqrt{a^{-2}-1}, \qquad u = \sqrt{\frac{x^2-a^2+1}{1-a^2}}$$

in (3), one obtains

$$\int_0^b \frac{\mathrm{d}x}{\sqrt{x^2-a^2+1}}\tan^{-1}\frac{\sqrt{x^2-a^2+1}}{a} = Ti_2\left(\frac{\sqrt{b^2+1-a^2}+b}{1+a}\right)$$
$$- Ti_2\left(\frac{\sqrt{b^2+1-a^2}-b}{1+a}\right). \qquad (4)$$

We next consider the integral

$$g(a,b) = \int_0^\infty \frac{\mathrm{d}s}{s^2+1}\ln\left[\sqrt{b^2(s^2+1)+1}+a\right]$$

for which it is elementary to determine

$$g(a,0) = \frac{\pi}{2}\ln(1+a), \qquad \frac{\partial}{\partial u}g(a,u) = \frac{\tan^{-1}\left(\sqrt{u^2+1-a^2}/a\right)}{\sqrt{u^2+1-a^2}}.$$

Therefore, by integrating over u using (4), we find

$$g(a,b) = \frac{\pi}{2}\ln(1+a) + Ti_2\left(\frac{\sqrt{b^2+1-a^2}+b}{1+a}\right) - Ti_2\left(\frac{\sqrt{b^2+1-a^2}-b}{1+a}\right)$$

which is easily transformed into

$$\int_0^\infty \frac{\ln\left[\alpha+\sqrt{\beta^2x^2+\gamma^2}\right]}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2}\ln\left[\beta+\sqrt{\gamma^2-\alpha^2}\right] + Ti_2\left(\frac{\alpha+\sqrt{\gamma^2-\beta^2}}{\beta+\sqrt{\gamma^2-\alpha^2}}\right) + Ti_2\left(\frac{\alpha-\sqrt{\gamma^2-\beta^2}}{\beta+\sqrt{\gamma^2-\alpha^2}}\right). \quad (5)$$

The argument of the logarithm in the integrand of $F(a,b)$ in (2) can be factored: let $R = \sqrt{[ac+b(1-b)]x^2+(1-b)(a+b)}$; then

$$(1-a)x^2+1+a+2R = [ac+b(1-b)]^{-1}[a(1-a)+2bc+(1-a)R][a+R].$$

The integrals resulting from inserting this into (2) are either elementary or can be evaluated by using (5). After some algebraic manipulation, we obtain

$$W(a,b) = 4\pi^2\ln\frac{d}{2} + 8\pi[Ti_2(a/d)+Ti_2(b/d)+Ti_2(c/d)] \quad (6)$$

where $d=\sqrt{ab+bc+ac}$, which is equivalent to Chen and Wu's formula.

In conclusion, we list a few values of the anisotropic triangular lattice Green function that can be obtained from (6) by differentiation. Here, $\Delta(a,b,c) = a+b+c-a\cos x-b\cos y-c\cos(x+y)$, $d=\sqrt{ab+bc+ca}$.

$$\int_0^{2\pi}\mathrm{d}x\int_0^{2\pi}\mathrm{d}y\,\frac{\Delta(1,0,0)}{\Delta(a,b,c)} = \frac{4\pi}{\mathrm{d}^2}(b+c)\left[\frac{\pi}{2}+\frac{a(b+c)+2bc}{a(b+c)}\tan^{-1}(a/d)-\tan^{-1}\left(\frac{(b+c)d}{bc+d^2}\right)\right]$$

$$\int_0^{2\pi}\mathrm{d}x\int_0^{2\pi}\mathrm{d}y\,\frac{\Delta(0,0,1)}{\Delta(a,b,c)} = \frac{4\pi}{d^2}(a+b)\left[\frac{\pi}{2}+\frac{(a+b)c+2ab}{(a+b)c}\tan^{-1}(c/d)-\tan^{-1}\left(\frac{(a+b)d}{ab+d^2}\right)\right]$$

$$\int_0^{2\pi}\mathrm{d}x\int_0^{2\pi}\mathrm{d}y\,\frac{\Delta(1,1,1)}{\Delta(a,b,c)} = \frac{4\pi^2}{d^2}(a+b+c) + \frac{8\pi}{d^2}\left[\frac{bc-a^2}{a}\tan^{-1}(a/d)+\frac{ac-b^2}{b}\tan^{-1}(b/d)+\frac{ab-c^2}{c}\tan^{-1}(c/d)\right]$$

$$\int_0^{2\pi}\mathrm{d}x\int_0^{2\pi}\mathrm{d}y\,\frac{\Delta(-1,1,0)}{\Delta(a,b,c)} = \frac{2\pi^2}{d^2}(a-b)+\frac{4\pi}{d^2}\left[(b-a)\tan^{-1}(c/d) + \left(a+b+\frac{2c(a+b)}{b}\right)\tan^{-1}(b/d) - \left(a+b+2c+\frac{2c(a+b)}{a}\right)\tan^{-1}(a/d)\right].$$

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